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Notes on Quaternionic Group Representations

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We study quaternionic group representations of finite groups systematically and obtain some basic tools of the theory, such as orthogonality relations and the Clebsch–Gordan series for reducible representations. We also derive all irreducible inequivalent Q-representations of a group G, classifying them according to a suitable generalization of the Wigner–Frobenius–Schur classification.

1. INTRODUCTION

Many attempts have been made to formulate quantum mechanics in vector spaces over the skew-field Q of the quaternions. In the early 1960s a series of papers (Finkelstein *et al.*, 1959, 1962, 1963) tried a systematic approach to quaternion quantum mechanics; these also constitute the starting point for a theory of quaternionic group representations (QGR).

Despite many considerable difficulties that we do not examine here, new efforts in this direction have been made in the last decade (Adler, 1986; Rotelli, 1989; De Leo and Rotelli, 1995), also resorting to group-theoretic techniques, as usual in complex quantum mechanics and quantum field theory.

We intend to study QGR directly (i.e., without the detour of transcribing the quaternion operators into complex ones via the symplectic representation) and systematically in this paper, going over the basic steps of the theory. We follow the spirit of the papers quoted above, using their suggestions and major results, but we limit our study to the case of *linear* representations of finite (or compact) groups.

When dealing with this subject, the main difficulties come from the noncommutativity of Q, which complicates from the very beginning the basic

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problem of the invertibility of a linear mapping, so that a new definition of determinant (which reduces to the usual one when applied to the complex case) is needed (Chen, 1991a,b) and the usual form of the character of a representation must be abandoned in favor of a (seemingly) weaker characterization. Moreover, since the field Q is not algebraically closed, the corollary of the Schur lemma (which is a basic tool for the analysis of representations and for deriving orthogonality relations) fails to be true in its usual form. This notwithstanding, we obtain in Section 3 (after having devoted Section 2 to recalling the basic notation and properties of the quaternion vector spaces) some orthogonality relations for linear representations and characters in QGR that can be applied to analyze any reducible Q-representation, so that we can prove that two Q-representations are equivalent if and only if they have the same (real) character. Furthermore, we obtain in Section 4 all the (inequivalent) irreducible Q-representations (Q-irreps) of a (finite) group G and classify them according to a generalization of the well-known Wigner–Frobenius–Schur classification of C-representations.

2. QUATERNION VECTOR SPACES

A quaternion is usually expressed as

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

where $q_i \in R$ ($i = 0, 1, 2, 3$), $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$.

The quaternion skew-field Q is an associative algebra of rank 4 over R, noncommutat' e, and endowed with an involutory antiautomorphism (*conjugation*) such that

$$q \to q^Q = q_0 - q_1 i - q_2 j - q_3 k$$

One can verify that

$$\forall p, q \in Q \qquad (pq)^Q = q^Q p^Q$$

Every quaternion q admits the so-called *symplectic decomposition* (Curtis, 1979)

$$q \to \begin{pmatrix} q_0 + q_1 i & -q_2 - q_3 i \\ q_2 - q_3 i & q_0 - q_1 i \end{pmatrix}$$

In a (right) n-dimensional vector space Q^n over Q, every linear operator is associated in a standard way (Curtis, 1979) to an $n \times n$ matrix acting on the left. Every quaternionic matrix admits a ($2n$-dimensional) *symplectic representation* which consists in substituting every matrix element by its symplectic decomposition (we note explicitly that the trace of every symplectic representation is necessarily real). Moreover, a mapping A is proved to be invertible (Chen, 1991b) whenever the so-called *double determinant* (i.e., $\det AA^{\dagger}$) is different from zero, and this condition is easily recognized (Scolarici, 1994) to coincide with the condition introduced, in a more intuitive fashion, by Finkelstein *et al.* (1959).

Finally, the relation

$$(\bar{x}, \bar{y}) = \sum_{i} x_i^Q y_i$$

(where x_i, y_i are the components in Q^n of the vectors $\bar{x}$, $\bar{y}$) defines a scalar product in Q^n.

In analogy with the case of complex group representations (CGR), one can then define the Hermitian conjugate $A^{\dagger} = A^{TQ}$ of a matrix A (A^T denotes, as usual, the transpose of A), and introduce the concepts of unitarity, Hermiticity, and so on. The properties of Hermitian and unitary matrices have been investigated (Finkelstein *et al.*, 1959, 1962, 1963); we stress here only the fact that, if G is a finite (or a compact) group, reducibility implies complete reducibility even in the case of unitary Q-representations $D(G)$, and every Q-representation is equivalent to a unitary one (the proofs follow trivially from those supplied in the complex case with minimal changes) (Scolarici, 1994).

Finally we recall that for Q-irreps the Schur lemma still holds (Finkelstein *et al.*, 1963), while its corollary must be generalized as follows:

"If a Hermitian matrix H commutes with an irreducible set D of matrices, it is a (real) multiple of the unit matrix" (Finkelstein *et al.*, 1963).

3. ORTHOGONALITY RELATIONS AND ANALYSIS OF Q-REPRESENTATIONS

Let $D(G)$ be an n-dimensional irreducible and unitary Q-representation of a finite group G and let us consider the matrix

$$A = \sum_{g \in G} D(g^{-1}) X D(g) = \sum_{g \in G} D^{TQ}(g) X D(g) \tag{1}$$

with X Hermitian; then, trivially, $A = A^{\dagger}$.

Indeed

$$A_{ij} = \sum_{g \in G} \sum_{k,l} D_{ki}^Q(g) X_{kl} D_{lj}(g) = A_{ji}^Q \tag{2}$$

Moreover, $D(g)A = AD(g)$, $\forall g \in G$.

By using the corollary of Schur's lemma (Finkelstein *et al.*, 1963), we obtain

$$A = \lambda^{(X)} I_n \tag{3}$$

where $\lambda^{(X)} \in R$ and I_n is the unit $n \times n$ matrix.

Let us choose now in (1) a matrix $X^{(r)}$ in such a way that $X^{(r)}_{kl} = \delta_{kr}\delta_{lr}$ with r fixed, and take the real trace of A. Recalling that the real trace satisfies the cyclic property Re Tr BC = Re Tr CB (Finkelstein *et al.*, 1963; Rotelli, 1989), we obtain

$$\text{Re Tr } A = \sum_g \text{Re Tr } X^{(r)} = [G] = \lambda^{(r)} n \tag{4}$$

where $[G]$ is the order of G.

By substituting the explicit form of $X^{(r)}_{kl}$ and $\lambda^{(r)}$ in equation (2), we easily obtain

$$\sum_{g \in G} D^{Q}_{ri}(g) D_{rj}(g) = \frac{[G]}{n} \delta_{ij} \tag{5}$$

Analogously, let $D^{(\mu)}(G)$ and $D^{(\nu)}(G)$ ($\mu \neq \nu$) be two unitary inequivalent Q-irreps of G whose dimensions, respectively, are n_μ and n_ν; then the matrix

$$A = \sum_{g \in G} D^{(\mu)}(g^{-1}) X D^{(\nu)}(g) \tag{6}$$

for every matrix X satisfies the condition

$$D^{(\mu)}(h) A = A D^{(\nu)}(h), \qquad \forall h \in G$$

By using the Schur lemma (Finkelstein *et al.*, 1963), we conclude that A must vanish identically.

Choosing in (6) a matrix $X^{(rs)}$ such that $X^{(rs)}_{kl} = \delta_{kr}\delta_{ls}$ with r, s fixed and writing down the explicit form of A_{ij}, we obtain

$$\sum_{g \in G} D^{(\mu)Q}_{ri}(g) D^{(\nu)}_{sj}(g) = 0 \tag{7}$$

and finally [expressing equations (5) and (7) in a more compact form],

$$\sum_{g \in G} D^{(\mu)Q}_{ri}(g) D^{(\nu)}_{rj}(g) = \frac{[G]}{n_\mu} \delta_{ij} \delta_{\mu\nu} \tag{8}$$

which is the (weaker) analog for Q-irreps of the orthogonality relation for C-irreps.

Let us now put $r = i$ and $s = j$ in equation (7), and let us sum over i and j; then,

$$\sum_g \chi^{(\mu)Q}(g) \chi^{(\nu)}(g) = 0 \tag{9}$$

where $\chi^{(\mu)}(g)$ denotes the (full) trace of $D^{(\mu)}(g)$. Equation (9) expresses the orthogonality between (quaternionic) characters of two inequivalent Q-irreps of the group G.

On the other hand, the following identity holds:

$$\hat{\chi}^{(\mu)}(g) \equiv \text{Re}\ \chi^{(\mu)}(g) = \frac{1}{4}\left[\chi^{(\mu)}(g) - i\chi^{(\mu)}(g)i - j\chi^{(\mu)}(g)j - k\chi^{(\mu)}(g)k\right]$$

and each term in parentheses, say $-i\chi^{(\mu)}(g)i$, can be considered as the character of g in a Q-representation (in our case $-iD^{(\mu)}i$), which is equivalent to the $D^{(\mu)}$, but certainly inequivalent to the $D^{(\nu)}$ (Scolarici, 1994). For, we easily get the following relation from (9):

$$\frac{1}{\sum_g \hat{\chi}^{(\mu)2}(g)} \sum_g \hat{\chi}^{(\mu)}(g)\hat{\chi}^{(\nu)}(g) = \delta_{\mu\nu} \tag{10}$$

or also (remembering that conjugated elements of a group have the same real character)

$$\frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i^{(\mu)} \hat{\chi}_i^{(\nu)} = \delta_{\mu\nu} \tag{11}$$

where $\hat{\chi}_i^{(\mu)}$ indicates obviously the (real) character of all elements belonging to the ith conjugation class of G, and k_i is the number of the elements of such a class.

As usual in CGR theory, equation (11) can be read as an orthogonality relation between vectors in a κ-dimensional space (where κ is the number of the conjugation classes of G), so that we finally obtain that the number r of inequivalent Q-irreps of G must satisfy the inequality

$$r \leq \kappa \tag{12}$$

We will see later that some groups exist for which strict inequality actually holds in QGR, while this does not occur in CGR. More important, we stress that the choice of characterizing any Q-representation by means of the real part of the trace (due to the necessity of maintaining the cyclic property of this quantity) does not eliminate any relevant information, as we will see in Section 4.

The possibility of decomposing any reducible Q-representation follows at once from these results. Indeed, let

$$D(G) = \sum_{\mu} a_{\mu} D^{(\mu)}(G)$$

be the Clebsch–Gordan series of a reducible Q-representation $D(G)$. Then, trivially,

$$\hat{\chi}(g) = \sum_{\mu} a_{\mu} \hat{\chi}^{(\mu)}(g) \qquad \forall g \in G$$

By using equation (11), we obtain

$$a_\mu = \frac{1}{\sum_i k_i \hat{\chi}_i^{(\mu)2}} \sum_i k_i \hat{\chi}_i \hat{\chi}_i^{(\mu)} \tag{13}$$

and this decomposition is unique, so that we can finally assert that *two Q-representations are equivalent if and only if their (real) characters coincide.*

4. *Q*-IRREPS AND THEIR CLASSIFICATION

We can now obtain and classify all irreducible *Q*-representations of a group *G*, generalizing the well-known classification of Wigner–Frobenius–Schur (Hamermesh, 1962; Dyson, 1962; Finkelstein *et al.*, 1963; Ascoli *et al.*, 1974; Garola and Solombrino, 1985). We briefly recall that a (unitary) *C*-irrep is said to be of class 0 if it is not equivalent to its complex conjugate, while it is said to be of class +1 (respectively, −1) if it is equivalent to its complex conjugate

$$D^*(g) = CD(g)C^{-1} \qquad \forall g \in G$$

and *C* is symmetric (respectively, antisymmetric). In the last cases, the (complex) character $\chi^{(\mu)}$ of $D(G)$ turns out to be real; moreover, the dimension is necessary even for representations of class −1.

Furthermore, it is well known that the following relation holds (Hamermesh, 1962):

$$\sum_g \chi^{(\mu)}(g^2) = c^{(\mu)}[G] \tag{14}$$

where

$$c^{(\mu)} = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \quad \text{if} \quad D^{(\mu)} \quad \text{is of class} \quad \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

Then, any *C*-irrep of a group *G* can obviously be considered as a (not necessary irreducible) *Q*-representation and an important theorem (Main Reduction Theorem) states that "a *C*-irrep *D* reduces over *Q* (into two equivalent *Q*-irreps D_1 and D_2) if and only if *D* belongs to the class −1" (Finkelstein *et al.*, 1963).

The Main Reduction Theorem permits us to conclude that any *C*-irrep $D^{(\mu)}(G)$ generates a *Q*-irrep $\hat{D}^{(\mu)}(G)$, which has the same dimension when it is of class 0 or +1 (in these cases $D^{(\mu)} = \hat{D}^{(\mu)}$) and halved dimension when it is of class −1.

Let $\chi_C^{(\mu)}$ be the (complex) character of the C-irrep $D^{(\mu)}$ and $\hat{\chi}^{(\mu)}$ the (real) character of the Q-representation $\hat{D}^{(\mu)}$; then, by using the previous discussion, one obtains

$$\hat{\chi}^{(\mu)} = \begin{cases} \chi_C^{(\mu)} \\ \mathrm{Re}\ \chi_C^{(\mu)} \\ \frac{1}{2}\chi_C^{(\mu)} \end{cases} \qquad \text{when} \quad D^{(\mu)} \text{ is of class } \begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{15}$$

As to the equivalence between the $\hat{D}^{(\mu)}$'s, we recall that *two inequivalent C-irreps share the same real part of the character if and only if they are complex conjugate of each other* (see Appendix). Then, one can easily prove the following:

All the Q-representations $\hat{D}^{(\mu)}$ so found are inequivalent to each other, with the exception of those generated by a pair of complex conjugated representations of class 0.

Let indeed $D^{(\mu)}(G)$ and $D^{(\nu)}(G)$ be two C-irreps of classes, say, $+1$ and 0, respectively. Then, their orthogonality reads

$$\sum_g \chi_C^{(\mu)*}(g)\chi_C^{(\nu)}(g) = \sum_g \chi_C^{(\mu)}(g)[\mathrm{Re}\ \chi_C^{(\nu)}(g) + i\ \mathrm{Im}\ \chi_C^{(\nu)}(g)] = 0$$

which implies [by using also equation (14)]

$$\sum_g \hat{\chi}^{(\mu)}(g)\hat{\chi}^{(\nu)}(g) = 0$$

i.e., recalling equation (10), $\hat{D}^{(\mu)}$ is inequivalent to $\hat{D}^{(\nu)}$.

One can prove in a similar way the other inequivalences stated by the theorem; in addition, one gets at once that two (C-inequivalent) representations of class 0 are equivalent in Q when they are complex conjugate [e.g., $D^*(g) = -jD(g)j$] and are certainly inequivalent in Q in the other cases, since they have then different real parts of the character.

We note explicitly that the proposition above assures us that the real part of the character fully identifies all the Q-irreps $\hat{D}^{(\mu)}$ so found.

We can now prove our major result.

Theorem. No Q-irrep exists besides those generated (in the sense of the Main Reduction Theorem) by the C-irreps (Scolarici, 1994).

Proof. Let a Q-irrep $\hat{D}^{(\alpha)}$ exist which is inequivalent to the $\hat{D}^{(\mu)}$'s generated by the C-irreps of a group G. Then, its character must satisfy the condition

$$\sum_g \hat{\chi}^{(\alpha)}(g)\hat{\chi}^{(\mu)}(g) = 0 \qquad \forall \mu \neq \alpha \tag{16}$$

Let us consider now the symplectic representation $D^{(\alpha)}_{\text{sym}}$ of $\hat{D}^{(\alpha)}$. Its character $\chi^{(\alpha)}_{\text{sym}}$ is necessarily real (see Section 2); moreover, $\chi^{(\alpha)}_{\text{sym}} = 2\hat{\chi}^{(\alpha)}$. Then, the coefficients of the Clebsch–Gordan series are in CGR theory

$$\begin{aligned} a^{(\alpha)}_{\mu} &= \frac{1}{[G]} \sum_{g} \chi^{(\mu)*}_{C}(g)\chi^{(\alpha)}_{\text{sym}}(g) \\ &= \frac{1}{[G]} \sum_{g} [\text{Re}\ \chi^{(\mu)}_{C}(g) - i\ \text{Im}\ \chi^{(\mu)}_{C}(g)]\chi^{(\alpha)}_{\text{sym}}(g) \end{aligned}$$

Since these coefficients must be real by definition, one immediately gets, recalling (15) and (16), $a^{(\alpha)}_{\mu} = 0$ $(\forall \mu)$. Thus, $D^{(\alpha)}_{\text{sym}}$ cannot exist and the theorem is proved.

As a corollary, we also obtain that *the real character* $\hat{\chi}$ *fully characterizes any Q-irrep*, as we anticipated in Section 3.

Furthermore, we obtain from the previous results, recalling (12), that

$$r \not< \kappa \tag{17}$$

if and only if the group G admits C-irreps belonging to the class 0.

We conclude with some final remarks that allow us to classify intrinsically (i.e., without resorting to their "generating" C-irreps) all the Q-irreps of a (finite) group G. First, we recall that for any C-irrep the following relation holds (Hamermesh, 1962):

$$\sum_{g} |\chi^{(\mu)}(g)|^2 = [G] \tag{18}$$

Then, substituting (15) in (18) and observing that for complex characters [see again the Appendix, equation (A.5)]

$$\sum_{g} (\text{Re}\ \chi^{(\mu)}(g))^2 = \sum_{g} (\text{Im}\ \chi^{(\mu)}(g))^2$$

we finally obtain

$$\sum_{g} \hat{\chi}^{(\mu)2}(g) = \frac{[G]}{c^{(\mu)}} \tag{19}$$

where

$$c^{(\mu)} = \begin{cases} 1 \\ 2 \\ 4 \end{cases} \quad \text{when } D^{(\mu)} \text{ belongs to the class } \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

(the above relation also allows one to simplify the orthogonality relation and the related formulas in Section 3).

We also recall that a C-irrep $D^{(\mu)}$ (hence, a Q-irrep $\hat{D}^{(\mu)}$) can be expressed in a suitable basis by real matrices if and only if it is of class $+1$ (Hamermesh, 1962; Finkelstein *et al.*, 1963; Ascoli *et al.*, 1974). On the other hand, a C-irrep also is a Q-irrep when it is of class $+1$ or 0, while it reduces to a (halved) Q-irrep when of class -1. Then, based on our above treatment we can assert that (Dyson, 1962; Ascoli *et al.*, 1974):

- $\hat{D}^{(\mu)}$ is a Q-irrep "potentially real" or of type R if and only if $\Sigma_g \hat{\chi}^{(\mu)2}(g) = [G]$ if and only if $D^{(\mu)}$ belongs to the class $+1$.
- $\hat{D}^{(\mu)}$ is a Q-irrep "potentially complex" or of type C if and only if $\Sigma_g \hat{\chi}^{(\mu)2}(g) = \frac{1}{2}[G]$ if and only if $D^{(\mu)}$ belongs to the class 0.
- $\hat{D}^{(\mu)}$ is a Q-irrep "(purely) quaternionic" or of type Q if and only if $\Sigma_g \hat{\chi}^{(\mu)2}(g) = \frac{1}{4}[G]$ if and only if $D^{(\mu)}$ belongs to the class -1.

Finally, we observe that all the results in this paper can be generalized (with minimal and obvious changes) to the case of compact groups; thus, some results about the Q-irreps of the 3-dimensional rotation group that are already known in the literature (Finkelstein *et al.*, 1959) can immediately be recovered in our general framework.

APPENDIX

Let D and D' be two inequivalent C-irreps of the group G, whose characters have the same real part. Irreducibility of both representations implies

$$\frac{1}{[G]} \sum_l k_l |\chi_l|^2 = 1 = \frac{1}{[G]} \sum_l k_l |\chi_l'|^2 \tag{A.1}$$

Let us put now $\chi_l = \alpha_l + i\beta_l$ and $\chi_l' = \alpha_l' + i\beta_l'$. Then, it follows from equation (A.1) that

$$\sum_l k_l \alpha_l^2 + \sum_l k_l \beta_l^2 = \sum_l k_l \alpha_l^2 + \sum_l k_l \beta_l'^2 = [G] \tag{A.2}$$

Let us observe that should all β_l vanish, also all β_l' would vanish, so that D and D', having the same character, would be equivalent. This contradicts the above assumption, so that neither the β_l nor the β_l' can all be zero. Moreover, since D is inequivalent to D', D to D^*, and D' to D'^*, we have

$$\sum_l k_l \chi_l^* \chi_l' = 0 \tag{A.3}$$

from which

$$\sum_l k_l \alpha_l^2 + \sum_l k_l \beta_l \beta_l' = 0 \tag{A.4}$$

and analogously

$$\sum_l k_l\alpha_l^2 - \sum_l k_l\beta_l^2 = 0 \qquad \text{(A.5a)}$$

$$\sum_l k_l\alpha_l^2 - \sum_l k_l\beta_l'^2 = 0 \qquad \text{(A.5b)}$$

By combining linearly the previous equations, we obtain

$$\sum_l k_l[\beta_l^2 + \beta_l'^2 + 2\beta_l\beta_l'] = \sum_l k_l[\beta_l + \beta_l']^2 = 0$$

Hence $\beta_l = -\beta_l'$ for all l.

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